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# NUMERICAL SOLUTION OF THE STEADY INCOMPRESSIBLE NAVIER–STOKES EQUATIONS FOR THE FLOW PAST A SPHERE BY A MULTIGRID DEFECT CORRECTION TECHNIQUE

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## SUMMARY

A nested non-linear multigrid algorithm is developed to solve the Navier–Stokes equations which describe the steady incompressible flow past a sphere. The vorticity–streamfunction formulation of the Navier–Stokes equations is chosen. The continuous operators are discretized by an upwind finite difference scheme. Several algorithms are tested as smoothing steps. The multigrid method itself provides only a first-order-accurate solution. To obtain at least second-order accuracy, a defect correction iteration is used as outer iteration. Results are reported for $Re = 50$, 100, 400 and 1000.

KEY WORDS Navier–Stokes equations Rigid sphere Finite difference schemes Accuracy Multigrid Defect corrections

## INTRODUCTION

A large number of studies have been published on the subject of momentum, heat and mass transport to a body of revolution. Clift *et al.*[1] give an extensive coverage of transport phenomena around spherical particles.

For $Re \geqslant 1$ the Navier–Stokes equations for the flow past a sphere have only numerical solutions. The earliest finite difference solution was reported by Jenson[2] for steady state flow at $Re = 5$, 10, 20 and 40. Using the same technique, other workers extended the work of Jenson to finer grids and higher $Re$.[3–6] In these studies a quasi-local relaxation with an empirical formula for the relaxation parameter was used.

An alternative approach was employed in Reference 7: the time-dependent problem was solved. The solution is continued until the flow is effectively steady. The vorticity is time-stepped forward directly. At every time step the streamfunction equation is relaxed until a convergence criterion is achieved. This process is repeated until the vorticity converges. Clift *et al.*[1] mention that this variant sometimes requires less computation than the iterative approach.

In the papers mentioned above the computational efforts are not presented in detail. It can be seen, however, that the problem is a hard one and, with the methods used, necessitates long computer times.

In the past decade the progress in computer hardware as well as in numerical algorithms has enabled attempts to be made towards the analysis and numerical solution of highly complex flow problems. Among the methods developed recently the multigrid (MG) method emerges as a very powerful one.

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Implemented in existing hydrodynamic codes, MG assures a reduction of the computational time by at least 50%.[8-10] In References 8–10 the velocity–pressure formulation of the Navier–Stokes equations is used. Other experiments with the same formulation are reported in References 11–13. An MG solution of the fourth-order equation of the streamfunction is presented in Reference 14.

Because in this study the vorticity–streamfunction formulation is used, we focus our attention on the papers in which the same formulation is employed. Wesseling and Sonneveld[15] and Mol[16] present results for the driven cavity problem and driven cavity and flow past a circular cylinder respectively. The Il'in discretization procedure is used. Their MG (the sawtooth cycle for linear systems) works as an inner iteration into a Newton procedure. The results can be viewed primordially of mathematical interest. The direct solution with the non-linear MG is made by Ghia *et al.*[17] for the driven cavity problem and by Fuchs[18] for the flow in channels. Ghia *et al.*[17] used an upwind discretization and Stone's SIP as smoother.

The Navier–Stokes equations belong to the class termed convection–diffusion equations. For such problems MG provides only a first-order-accurate solution. The reason is explained in detail in Reference 14. To retain the advantages offered by MG and to obtain more accurate results, several techniques were proposed in Reference 14. Among these the most suitable for a general problem seems to be the defect correction (DC) iteration.[19,20] In the domain of the Navier–Stokes equations, Khosla and Rubin[21] were among the first to employ the combination of the preconditioned conjugate gradient (PCG) technique and DC. Ghia *et al.*[17] changed the PCG in the Khosla and Rubin algorithm to MG. Other similar experiments were done by Brandt[13] and Fuchs and Zhao[12].

The present study represents an effort to employ the combination MG–DC in the solution of the Navier–Stokes equations which describe the steady incompressible flow past a rigid sphere. Some new and interesting results are obtained. A nested DC–MG strategy—used to our knowledge for the first time in this paper—seems to be very efficient.

## MODEL EQUATIONS

Consider a rigid spherical particle of radius $a$ moving into an unbounded volume of fluid. The following assumptions are made.

(a) The flow is steady, axisymmetric and laminar.
(b) The physical properties of the particle and medium are constant.
(c) The shape and volume of the particle are constant.
(d) The fluid is Newtonian.

With these assumptions the Navier–Stokes equations in dimensionless form are

*streamfunction* ($\psi$)

$$\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial\theta^2}-\frac{\partial\psi}{\partial z}-\cot\theta\frac{\partial\psi}{\partial\theta}=\zeta\exp(3z)\sin\theta, \tag{1a}$$

*vorticity* ($\zeta$)

$$\frac{Re}{2}\left(\frac{\partial}{\partial\theta}(e^z V_\theta\zeta)+\frac{\partial}{\partial z}(e^z V_R\zeta)\right)=\frac{\partial^2\zeta}{\partial z^2}+\frac{\partial^2\zeta}{\partial\theta^2}+\frac{\partial\zeta}{\partial z}+\cot\theta\frac{\partial\zeta}{\partial\theta}-\frac{\zeta}{\sin^2\theta}, \tag{1b}$$

where

$$V_R = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad V_\theta = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial z}$$

and $Re$ is the particle Reynolds number.

The boundary conditions considered are

$$\theta = 0, \pi: \quad \psi = 0, \qquad \zeta = 0, \tag{1c}$$

$$z = 0: \quad \psi = 0, \quad \frac{\partial\psi}{\partial z} = 0, \quad \zeta = \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial z^2}, \tag{1d}$$

$$z = \infty: \quad \psi = 0{\cdot}50\exp(2z)\sin^2\theta, \quad \zeta = 0. \tag{1e}$$

In equations (1) the well known transformation $r = \exp z$ was applied.

## METHOD OF SOLUTION

Some difficulties of the MG solution of equations (1) originate from the discretization process. Therefore we begin with the discretization of equations (1).

For relatively large $Re$ in the equations (1) the convection dominates the diffusion. Discretization with the usual schemes of order at least two in accuracy leads in this case to an unstable discrete operator. One remedy is to use a local relaxation technique.[22] This alternative is discussed elsewhere.[23] Another remedy is to choose an adequate discretization scheme. The most usual schemes recommended in this case are presented in Reference 24. In the present paper an upwind finite difference scheme was used for equation (1b). Equation (1a) was discretized with a centred second-order-accurate scheme. Results with an artificial viscosity scheme are presented in Reference 25.

The discretization of the Navier–Stokes equations leads to a non-linear system of equations. In a first attempt these will be solved by an MG method. A complete description of the MG ideas and algorithms can be found in References 14, 26 and 27. Here only the major features of the MG algorithm are presented.

In the MG method one attempts to solve the discrete approximation

$$N_L u_L = F_L \tag{2}$$

to a continuous equation

$$Nu = F \tag{3}$$

on a sequence of grids $G_1, \ldots, G_L$ with corresponding mesh sizes $h_1 > \ldots > h_L$. To define the MG iteration step (cycle) the following operators must be specified:

*the non-linear grid operator*

$$N_k : \mathcal{G}(G_k) \to \mathcal{G}(G_k),$$

*the prolongation operator*

$$I_{k-1,k} : \mathcal{G}(G_{k-1}) \to \mathcal{G}(G_k),$$

*the restriction operator*

$$I_{k,k-1} : \mathcal{G}(G_k) \to \mathcal{G}(G_{k-1}),$$

*the iteration operator*

$$S_k: \mathcal{G}(G_k) \rightarrow \mathcal{G}(G_k),$$

where $\mathcal{G}(G_k)$ denotes the space of gridfunctions on $G_k$. The variant which is used in this paper is the nested FAS algorithm[26] suitable for general non-linear problems. In quasi-PASCAL the FAS step can be written as

procedure FAS($l$, $u_l$, $F_l$); integer $l$; array $u_l$, $F_l$ (4a)

if $l=1$ then $u_l := S_l^{\nu_1}(u_l, F_l)$ else (4b)

begin integer $i$; array $v$, $r$, $\tilde{u}$ (4c)

$u_l := S_l^{\nu_1}(u_1, F_l)$; $\tilde{u}_{l-1} := \text{Inj} * u_l$ (4d)

$r_{l-1} := I_{l,l-1} * (F_l - N_l u_l) + N_{l-1}\tilde{u}_{l-1}$ (4e)

$v_{l-1} := \tilde{u}_{l-1}$; for $i := 1(1)Y$ do FAS($l-1$, $v_{l-1}$, $r_{l-1}$) (4f)

$u_l := u_l + I_{l-1,l} * (v_{l-1} - \tilde{u}_{l-1})$ (4g)

$u_l := S_l^{\nu_2}(u_l, F_l)$ (4h)

end (4i)

where $Y=1$ for the V-cycle and $Y=2$ for the W-cycle.

It is well known[15,17,21] that successive relaxation of the individual equations (1a) and (1b) is an inefficient process and, especially in an MG algorithm,[17] has a poor smoothing rate. In this work, collective relaxation of equations (1a) and (1b) is used.

Point Gauss–Seidel (PGS), symmetric Gauss–Seidel (SGS), alternating line Gauss–Seidel (ALGS), incomplete lower–upper decomposition (ILU(1, 1)) and combinations of these were used as smoothers. The results concerning the smoothing efficiency of the methods mentioned above will be presented in the next section.

When a grid was visited for the first time, cubic interpolation[14] was used. In other cases the linear nine-point prolongation operator was employed. As a restriction, the experiments have been done with injection and full weighting. In (4d), Inj means injection. Four levels were used with $\nu_1=2$ and $\nu_2=1$. The coarsest has $9\times 9$ points with $h_z=h_\theta=\pi/8$; the finest has $65\times 65$ points with $h_z=h_\theta=\pi/64$.

Only with the MG method can a first-order-accurate solution (owing to the upwind discretization) be obtained. As can be seen in the next section, this solution is unacceptable from the point of view of accuracy. From the theoretical and experimental data presented in the literature,[1] second-order accuracy is at least desirable. The method employed in this work is to obtain high accuracy without solving a high-order discretization equation. In MG this can be done with the following strategies:[14]

(a) $\tau$-extrapolation
(b) Richardson extrapolation
(c) double discretization
(d) DC iteration.

The two extrapolation methods can be well used if the solution is indeed smooth. A drawback is that these methods rely on the existence of an asymptotic expansion of the truncation error for $h\rightarrow 0$, and globally, no *a priori* information about the validity of this assumption is available. In a similar situation for the Euler equations, Hemker[28] chooses DC iteration. Thus, in spite of the fact

that it is more expensive than the other methods, DC iteration seems to be the most adequate for a general problem.

If we denote by $N_{h,1}$ the upwind discretization of the non-linear operator $N$ and by $N_{h,2}$ the second-order discretization of $N$, the DC iteration can be written as[20]

$$N_{h,1}u_h^{i+1} = N_{h,1}u_h^i - [N_{h,2}u_h^i - F_h]. \tag{5}$$

It can be proved[20] that a solution of

$$N_{h,2}u_h = F_h \tag{6}$$

is also a fixed point of (5). One step of the DC iteration is not necessary to be effectively solved. Some iterations with a given iterative method are sufficient. In this work the DC step comprises one or more MG cycles. Taking into consideration the DC iteration too, the algorithm can be written as

start with an approximation for $u_1$ (7a)

for $l := 1(1)L-1$ do (7b)

begin (7c)

  for $i := 1(1)$iter do FAS($l$, $u_l$, $F_l$) (7d)

  $u_{l+1} := I^3_{l,l+1}u_l$ (7e)

end (7f)

$l = L$ (7g)

for $i := 1(1)$iter do FAS($L$, $u_L$, $F_L$) (7h)

for $j := 1(1)$ncdc do (7i)

begin (7j)

  $d := N_{L,1}u_L - N_{L,2}u_L + F_L$ (7k)

  for $i := 1(1)$kd do FAS($L$, $u_L$, $d$) (7l)

end (7m)

where iter = 15 for $l = 1$, iter = 2 for $1 < l \leqslant L$ and $I^3_{l,l+1}$ is the cubic interpolation. The quantity $d$ in (7k) is computed only for the vorticity.

In all the computations the initial values for $\psi$ and $\zeta$ were set equal to zero on all grid points except for $\psi$ at $z = \infty$ where the boundary condition (1e) holds.

## NUMERICAL RESULTS AND DISCUSSIONS

At the beginning of this section the behaviour of the MG algorithm (steps (7a)–(7h)) is discussed. The first aspect presented is the convergence property of the algorithm as a function of the smoothing method. The experiments done are summarized in Table I. Table I presents the results for $Re = 50$ and 1000 only. For $Re = 100$ and 400 the values obtained for $\bar{\rho}$ lie between those presented in Table I. Except for some surprises (the divergence of ILU at $Re = 50$ and the good comportment of PGS at $Re = 1000$), the values obtained for $\bar{\rho}$ are higher (but not significantly so) than those reported in the literature for standard linear problems. The time reported in Table I is the CPU time needed on a FELIX C512 computer in double-precision arithmetic for steps

Table I. Effect of the smoothing algorithm on the convergence of MG

| Smoother | | | | |
|---|---|---|---|---|
| Equation (1a) | Equation (1b) | *Re* | $\bar{\rho}$† | Time (s) |
| PGS | PGS | 50 | 0·251 | 263·28 |
| | | 1000 | 0·156 | |
| PGS | SGS | 50 | 0·224 | 407·33 |
| | | 1000 | 0·154 | |
| SGS | SGS | 50 | 0·291 | 430·0 |
| | | 1000 | 0·150 | |
| PGS | ILU | 50 | 0·229 | 428·79 |
| | | 1000 | 0·117 | |
| ILU | ILU | 50 | >1 | 462·50 |
| | | 1000 | 0·209 | |
| PGS | ALGS | 50 | 0·217 | 511·86 |
| | | 1000 | 0·112 | |
| ALGS | ALGS | 50 | 0·078 | 625·51 |
| | | 1000 | 0·047 | |

† $\bar{\rho}$, average reduction factor.

(7a)–(7h) of the algorithm presented in the preceding section. The method chosen is PGS + PGS. If we make a balance between $\bar{\rho}$ and time we see that PGS + PGS is a good choice. Concerning the use of injection or FW as restriction operator, the tests done show that FW is preferable.

The results of engineering interest usually reported in the literature for the subject of this paper are

(a) the surface vorticity
(b) the surface pressure
(c) drag coefficients
(d) vortex length.

The surface pressure and the drag coefficients were calculated using the following relations:[7,29]

*surface pressure*

$$P_s(\theta) = P_0 + \frac{4}{Re}\int_0^{\theta}\left(\frac{\partial\zeta}{\partial z}+\zeta\right)\Big|_{z=0} d\theta, \tag{8}$$

where

$$P_0 = 1 + \frac{8}{Re}\int_0^{\theta}\left(\frac{\partial\zeta}{\partial\theta}\right)\Big|_{\theta=0} dz, \tag{9}$$

*form drag coefficient*

$$C_{DP} = \int_0^{\pi} P_s \sin(2\theta) d\theta, \tag{10}$$

*skin friction drag coefficient*

$$C_{DF} = \frac{8}{Re}\int_0^{\pi} \zeta_s \sin^2\theta \, d\theta, \tag{11}$$

*total drag coefficient*

$$C_D = C_{DP} + C_{DF}. \tag{12}$$

The surface vorticity obtained with the MG algorithm is depicted in Figure 1 for $Re = 100$ only. The surface vorticity obtained with the artificial viscosity scheme[25] and the second-order-accurate solution are also depicted in this figure. The results at $Re = 50$, 400 and 1000 are similar, but it must be noted that the differences between the three solutions increase with an increase in $Re$. Using the number of cycles presented in the preceding section, the solution is obtained in the limit of the truncation error. Increasing the number of MG cycles (e.g. doubling them) changes the results (surface vorticity, drag coefficients) insignificantly. Figure 1 shows that the surface vorticity obtained with the first-order-accurate schemes is far from the second-order-accurate solution. The

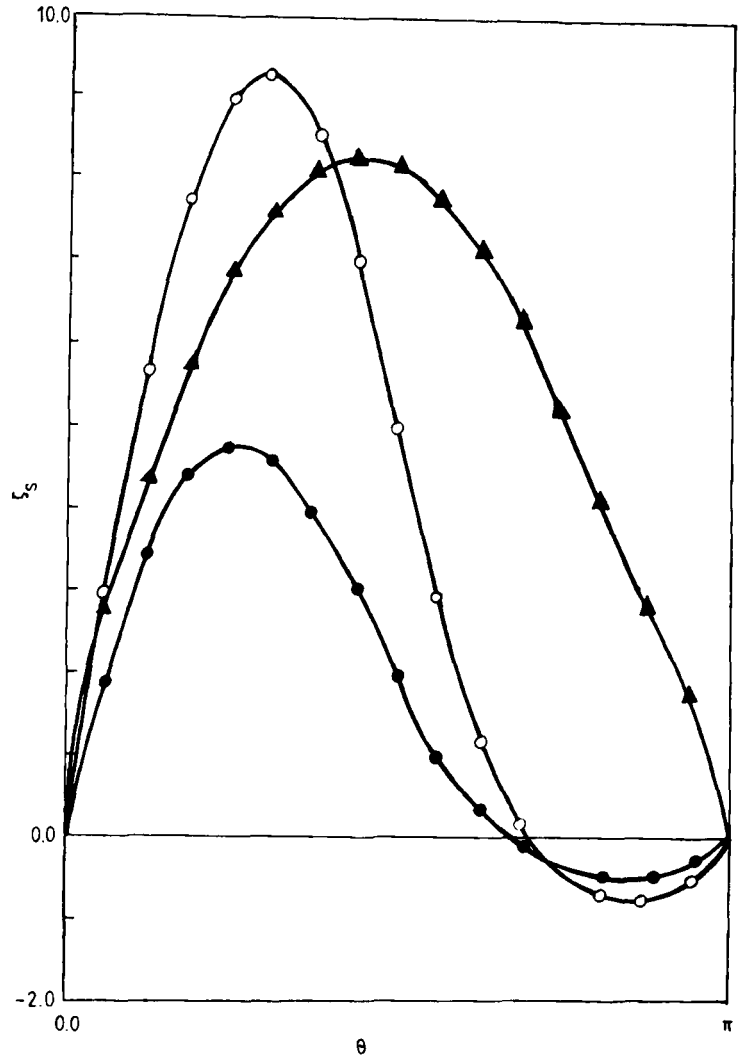


Figure 1. Surface vorticity at $Re = 100$: ▲, upwind; ●, artificial viscosity; ○, second-order-accurate solution

values of the surface pressure, drag coefficients and vortex length obtained with the first-order-accurate schemes are not presented, but the disagreement between these results and the second-order-accurate results is as large as in the case of the surface vorticity. In Reference 1 (p. 106) it can be seen that the results (separation angle and vortex length) of the second-order-accurate scheme were confirmed experimentally. Thus, from Figure 1, the necessity to improve the first-order-accurate solution is evident. As was mentioned in the preceding section, DC iteration was used to do this.

Hemker[28] presents two working alternatives valid for algorithm (7) if the DC iteration starts with a first-order initial approximation and second-order accuracy is desired. In the first it is sufficient to take ncdc = 1 and kd = $O(\log(h))$. In the other alternative kd = 1 and a sufficiently large number of ncdc is required. The choice between these two alternatives is made a function of the computational cost of the FAS step and the right-hand-side evaluation (the quantity $d$ in (7k).

For the DC iteration the following aspects will be analysed:

(a) the effect of the number of MG cycles used in the DC step on the convergence rate of the DC iteration
(b) the comparison between the convergence rate of the DC iteration with MG and the DC iteration with the corresponding single-grid algorithm
(c) the evolution of the solution during the DC process
(d) the evolution of the error between the solutions of two consecutive DC steps.

For the problems mentioned above all the tests were made at $Re = 50$ with a three-level algorithm (the finest level has only $33 \times 33$ points). The use of a four-level algorithm does not change anything in the results obtained. On increasing $Re$ the situation is similar but more iterations are required to reach the second-order-accurate solution.

The effect of the number of MG cycles used in the DC step on the convergence rate of the DC process is presented in Figures 2 and 3. In Figure 2 the behaviour of the total drag as a measure of the second-order accuracy is presented. In Figure 3 the error

$$\max_{i,j} |\psi_{i,j}^{n+1}(\zeta_{i,j}^{n+1}) - \psi_{i,j}^{n}(\zeta_{i,j}^{n})|, \tag{13}$$

where the superscript stands for the DC iteration, is presented. Experiments were also done using more than three MG cycles, but no improvement was observed. Also in these figures the results obtained with PGS + PGS in the single-grid version are presented. Four relaxation sweeps, which are equivalent to one MG cycle, are made in the DC step. Experiments were also done with eight and 12 relaxation sweeps in the DC iteration, but these results are not depicted in order to alleviate clutter in Figures 2 and 3. No information is lost as a result of this. From Figures 2 and 3 the following conclusions can be drawn.

(a) An increase of the number of MG cycles in the DC step accelerates the convergence rate of the DC process only in the initial stage.
(b) The most efficient version is that with one MG cycle.
(c) The MG and single-grid results are close to each other.
(d) The first alternative of Hemker[28] does not work in this case.
(e) The idea, widely used in the literature, that a small number of DC iterations are necessary is not valid.
(f) A clear correspondence between the error defined in (13) and the accuracy of the total drag cannot be made; this correspondence is function of the number of MG cycles used in the DC step.

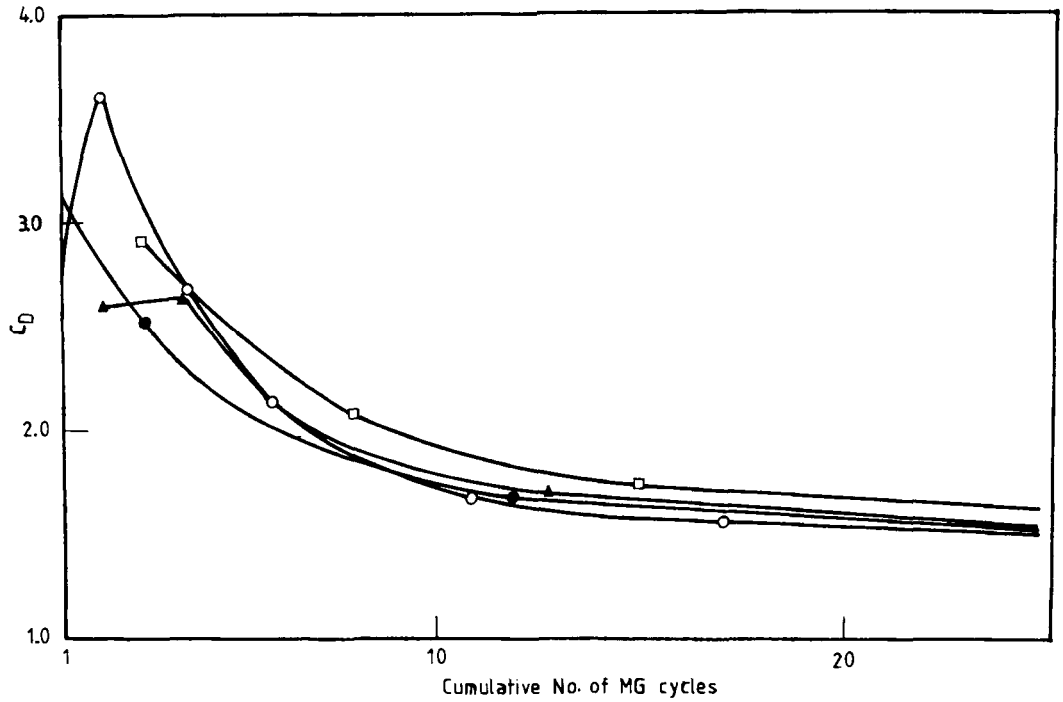


Figure 2. Evolution of the total drag coefficient during the DC process as a function of the number of MG cycles: ○, one MG cycle; ▲, two MG cycles; □, three MG cycles; ●, SG

The statements made previously concerning the effect on the convergence of the DC iteration of the number of MG cycles used in the DC step and the behaviour of the single-grid version can be previsedfrom the proposition of Auzinger and Stetter (Reference 30, pp. 330–331).

The average reduction factor of the DC process for all variants presented in Figures 2 and 3 can be viewed in Table II. As can be seen, the combination DC–MG is a relatively slowly convergent iteration. To complete the description, the evolutions of the surface vorticity, surface pressure and vortex length during the DC process are presented in Figures 4–6 respectively. These results were obtained with the four-level algorithm.

Another aspect of the DC process is the convergence criterion. A generally valid convergence criterion is not yet established in the literature. Auzinger and Stetter[30] and Auzinger[20] for a weakly non-linear diffusion problem, Hemker[31] for a standard linear convection–diffusion equation and Hemker[28] and Spekreijse[32] for Euler equations do not present a clear criterion for the convergence of the DC process. The iteration is said to be convergent when the DC solution attains the high-order-accurate solution without any numerical specification. In Reference 17 the norm of the inner iteration residuals is depicted as the convergence measure. In the present study the DC iteration is said to be convergent when the first three significant figures of the maximum surface vorticity and its location, of the total drag coefficient and of the vortex length remain unchanged. Our criterion corresponds to a value of $\sim 2 \times 10^{-3}$ for the error defined in (13). Note that this correspondence is conditioned by the number of MG cycles (one in this case) used in a DC step.

The results obtained with this error criterion are in good agreement with those reported in the literature. The numerical values of the separation angle, vortex length and drag coefficients are

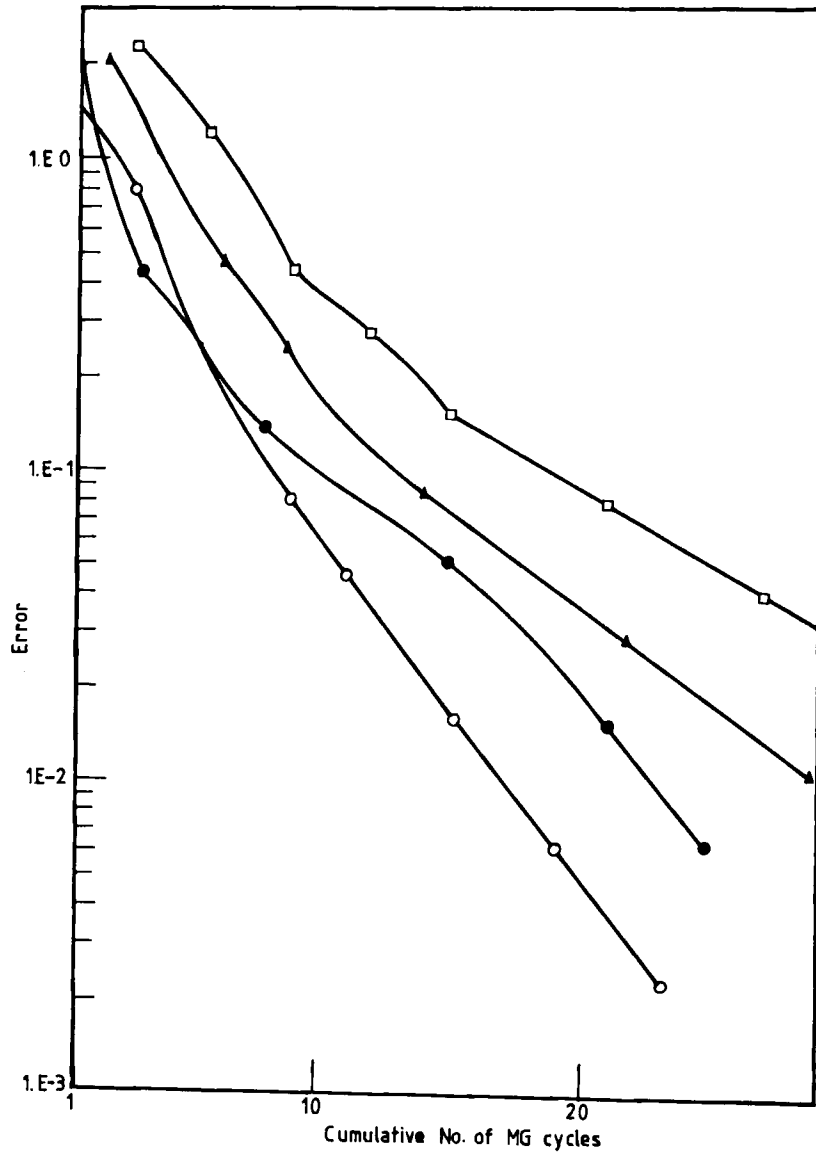


Figure 3. Evolution of the DC iteration error as a function of the number of MG cycles: ○, one MG cycle; ▲, two MG cycles; □, three MG cycles; ●, SG

Table II. Average reduction factor for the different variants of the DC process

| Variant | Number of inner iterations | $\bar{\rho}$ |
|---|---|---|
| DC-MG | 1 | 0·76 |
| DC-MG | 2 | 0·742 |
| DC-MG | 3 | 0·741 |
| DC-SG | 4 | 0·80 |

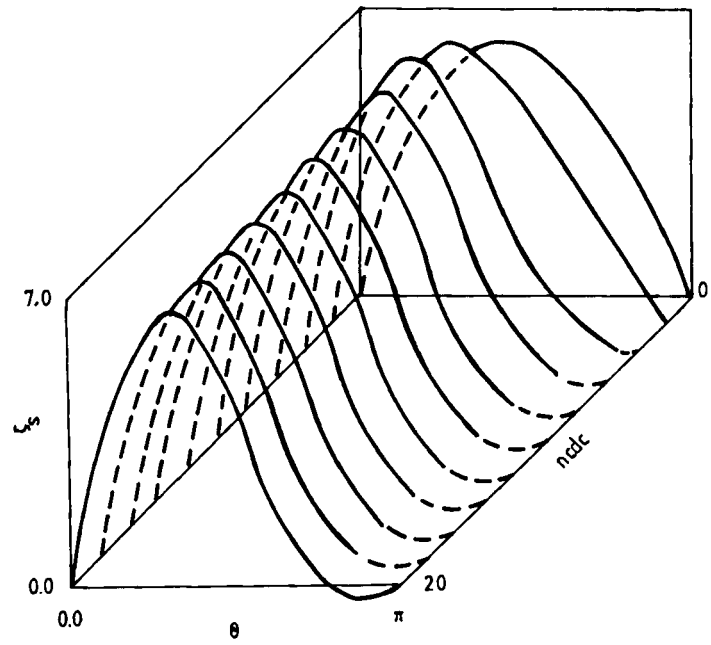


Figure 4. Evolution of the surface vorticity during the DC iteration at $Re = 50$

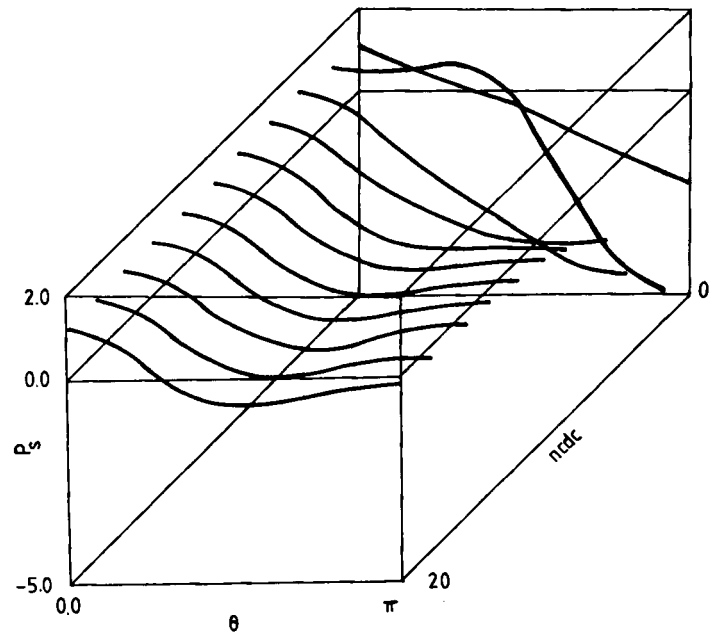


Figure 5. Evolution of the surface pressure during the DC iteration at $Re = 50$

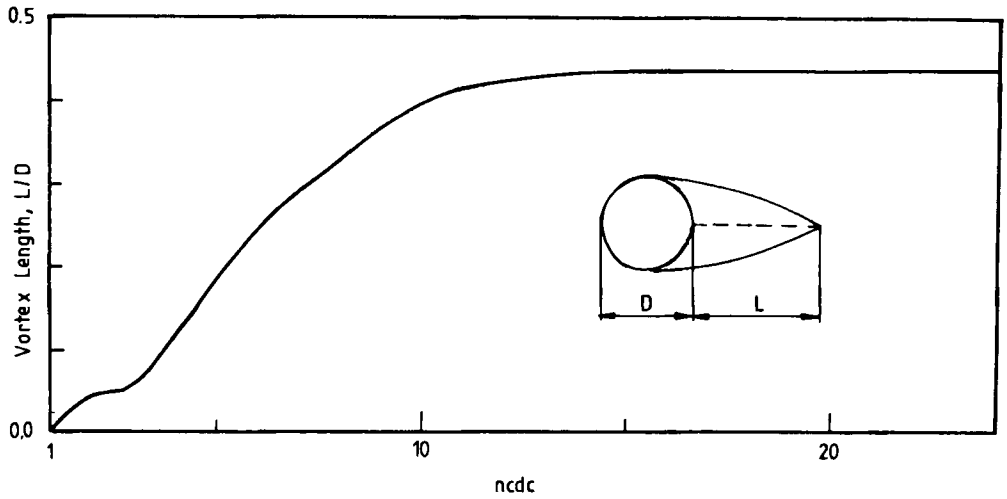


Figure 6. Evolution of the vortex length during the DC iteration at $Re = 50$

Table III. Numerical values for drag coefficients, vortex length and separation angle

| | Reference 29 | | | Reference 1 | | Present work | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $C_{DP}$ | $C_{DF}$ | $C_D$ | $L/D$ | $\theta_s$ | $C_{DP}$ | $C_{DF}$ | $C_D$ | $L/D$ | $\theta_s$ |
| 50 | 0·661 | 0·915 | 1·576 | 0·45 | 140 | 0·639 | 0·917 | 1·556 | 0·45 | 140 |
| 100 | 0·512 | 0·576 | 1·088 | 0·93 | 127 | 0·493 | 0·576 | 1·069 | 0·93 | 127 |

Table IV. Number of DC iterations necessary to attain the error criterion

| *Re* | Number of DC steps |
|---|---|
| 50 | 25 |
| 100 | 38 |
| 400 | — |
| 1000 | Diverging |

depicted in Table III together with the values presented in References 1 and 29. As can be seen, the concordance in drag coefficients is better than 5%. The separation angle and the vortex length coincide with those of Reference 1.

In Table IV the number of DC iterations with one MG cycle as inner iteration necessary to attain the convergence criterion is presented. As can be seen, the DC iteration converges only at $Re = 50$ and 100. At $Re = 400$ the DC iteration becomes oscillatory and at $Re = 1000$ it diverges. With the artificial viscosity scheme of Reference 25 the DC process converges at $Re = 400$. In spite of the fact that the convergence criterion used in this work is not too severe, a relatively large number of DC iterations compared with References 28 and 30–32 are necessary. To improve the

performance of the DC–MG iteration, the following strategy, which can be viewed as a nested DC–MG algorithm, is proposed:

start with an approximation for $u_1$ (14a)

$l := 1$; for $i := 1(1)15$ do FAS($l$, $u_l$, $F_l$) (14b)

$u_{l+1} := I^3_{l,l+1} u_l$ (14c)

for $l := 2(1)L$ do (14d)

begin (14e)

for $j := 1(1)$ ncdc $(l)$ do (14f)

begin (14g)

$d_l := N_{l,1} u_l - N_{l,2} u_l + F_l$ (14h)

for $i := 1(1)$ kd $(j)$ do FAS($l$, $u_l$, $d_l$) (14i)

end (14j)

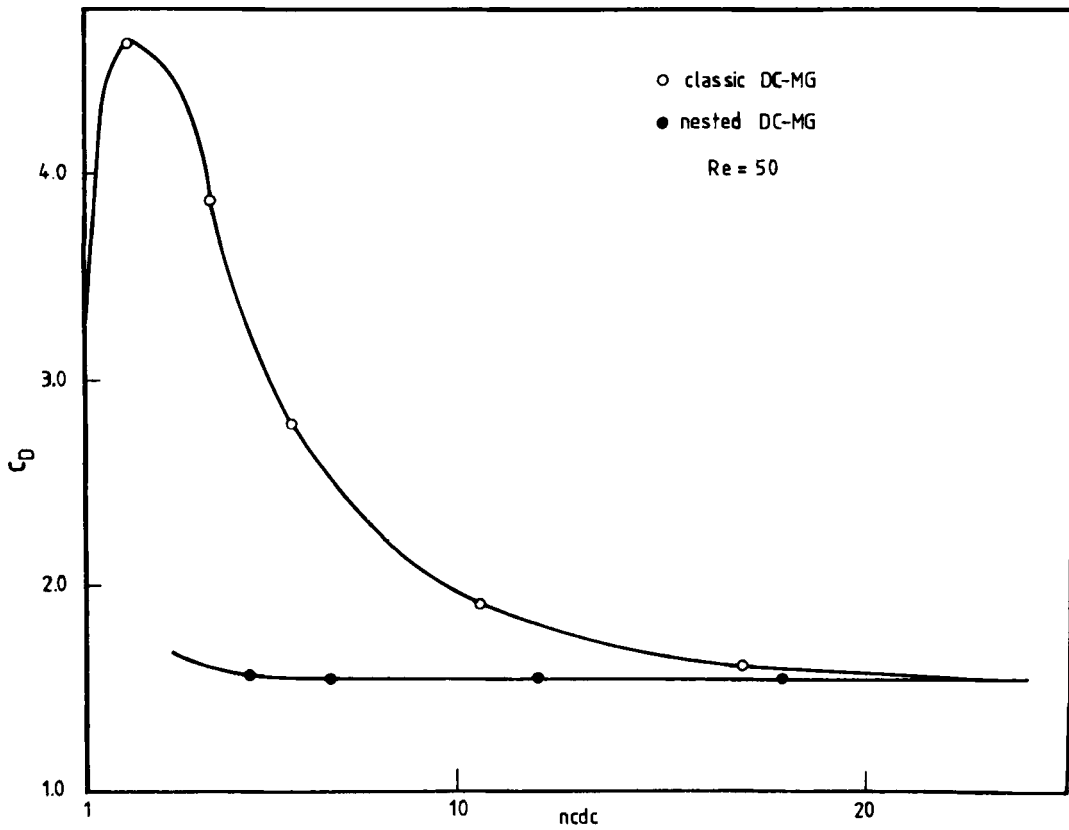


Figure 7. Comparison in terms of the total drag coefficient between the classical and nested DC–MG iterations

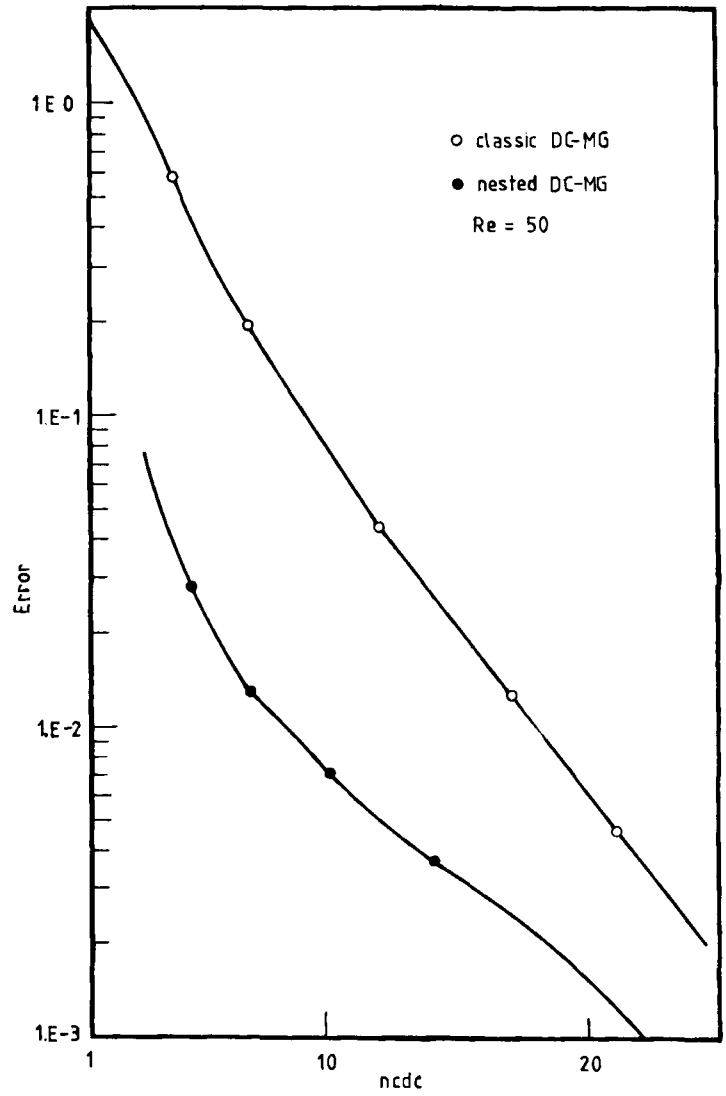


Figure 8. Comparison in terms of the DC iteration error between the classical and nested variants of the DC–MG process

if $l<L$ then $u_{l+1}:=I^3_{l,l+1}u_l$ else (14k)

end (14l)

where kd(1)=2 and kd($j$)=1 for $j>1$.

A comparison between the classical and nested variants of the DC–MG process is made in Figures 7 and 8. For the nested variant, since the work on the previous levels is taken into account, the corresponding curves are displaced to the right. As can be seen, the improvement is considerable. On the finest level (level 4) only five iterations at $Re=50$ and 10 iterations at $Re=100$ are needed to reach the error criterion. The problem for the nested variant is the number of DC steps on levels 2 and 3. The following variants were found to be quasi-optimal:

(a) at $Re = 50$, 15 DC iterations on level 2, five DC iterations on level 3
(b) at $Re = 100$, 20 DC iterations on level 2, eight DC iterations on level 3.

As can be seen, in the nested variant too the number of DC iterations on the finest level depends on $Re$ and increases with an increase in $Re$.

In References 1 and 3–7, sufficient information is not presented on the computational efficiency in order to make a fair comparison with the method used in this work. In addition, the following statements can be made.

(a) The number of relaxation steps in the quasi-local relaxation methods is considerably larger than for the DC–MG algorithm; in spite of the fact that a DC–MG step consumes at least three times more computer time, DC–MG (in the classical variant) is globally faster than the quasi-local relaxation methods.
(b) The quasi-local relaxation procedures are very sensitive to the value of the relaxation factor for the boundary condition (1d) (for vorticity); this causes trouble in using the relaxation methods in the nested variant.
(c) If a time step in the transient formulation is considered quasi-equivalent to a DC step, the number of DC steps is less than the number of time steps used in Reference 7.

## CONCLUSIONS

An efficient calculation procedure for the steady state Navier–Stokes equations in the $\psi$–$\zeta$ formulation has been developed using the defect correction technique with MG as inner iteration. For the flow past a sphere the solution obtained with the first-order-accurate discretization schemes is far from the physical solution. At least second-order accuracy in the solution is desired. The method used in this study to obtain the second-order-accurate solution is DC iteration. As inner iteration in the DC step the MG method was used. The most efficient version of the DC–MG algorithm seems to be that with one MG cycle (V-cycle) in a DC step. Also the DC method with the smoothing step of the MG algorithm used in the single-grid version as inner iteration performs very close to the DC–MG version. Compared with the other cases solved with the DC–MG algorithm, the flow past a sphere necessitates a relatively large number of DC steps (up to 40). Using the nested variant of the DC–MG algorithm the computational effort is reduced by 60%.

Starting with the zero approximation the DC–MG algorithm converges to the desired solution for $Re$ up to 400. The quasi-local elaxation methods converge in the same domain of $Re$. The transient formulation provides steady solutions for $Re$ up to 1000.

Compared with the previous methods used to solve the steady flow past a sphere, we can conclude that the DC–MG method is faster.

## APPENDIX: NOMENCLATURE

| | |
|---|---|
| $a$ | sphere radius |
| $F$ | streamfunction |
| $h_\theta$ | lattice spacing in the $\theta$-direction |
| $h_z$ | lattice spacing in the $z$-direction |
| $p$ | pressure |
| $P$ | dimensionless pressure, $(p - p_\infty)/0{\cdot}50\rho U^2$ |
| $r$ | dimensionless radial distance, $R/a$ |
| $R$ | radial distance |

| | |
|---|---|
| $Re$ | Reynolds number, $2U/v$ |
| $U$ | free stream velocity |
| $V_R$ | radial velocity component |
| $V_\theta$ | tangential velocity component |
| $Y$ | vorticity |
| $z$ | dimensionless radial distance |

*Subscripts*

| | |
|---|---|
| $i$ | defining mesh point in the $\theta$-direction |
| $j$ | defining mesh point in the $z$-direction |
| s | evaluated at $z=0$ |

*Superscripts*

| | |
|---|---|
| $n$ | DC iteration number |

*Greek*

| | |
|---|---|
| $\zeta$ | dimensionless vorticity, $Ya/U$ |
| $\theta$ | polar angle, spherical co-ordinate system |
| $v$ | kinematic viscosity |
| $\rho$ | density |
| $\psi$ | dimensionless streamfunction, $F/\rho Ua^2$ |

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